VSB TECHNICAL UNIVERSITY OF OSTRAVA | FACULTY OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE | DEPARTMENT OF COMPUTER SCIENCE

Data Visualization

460-4120

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Tensor Data Visualization

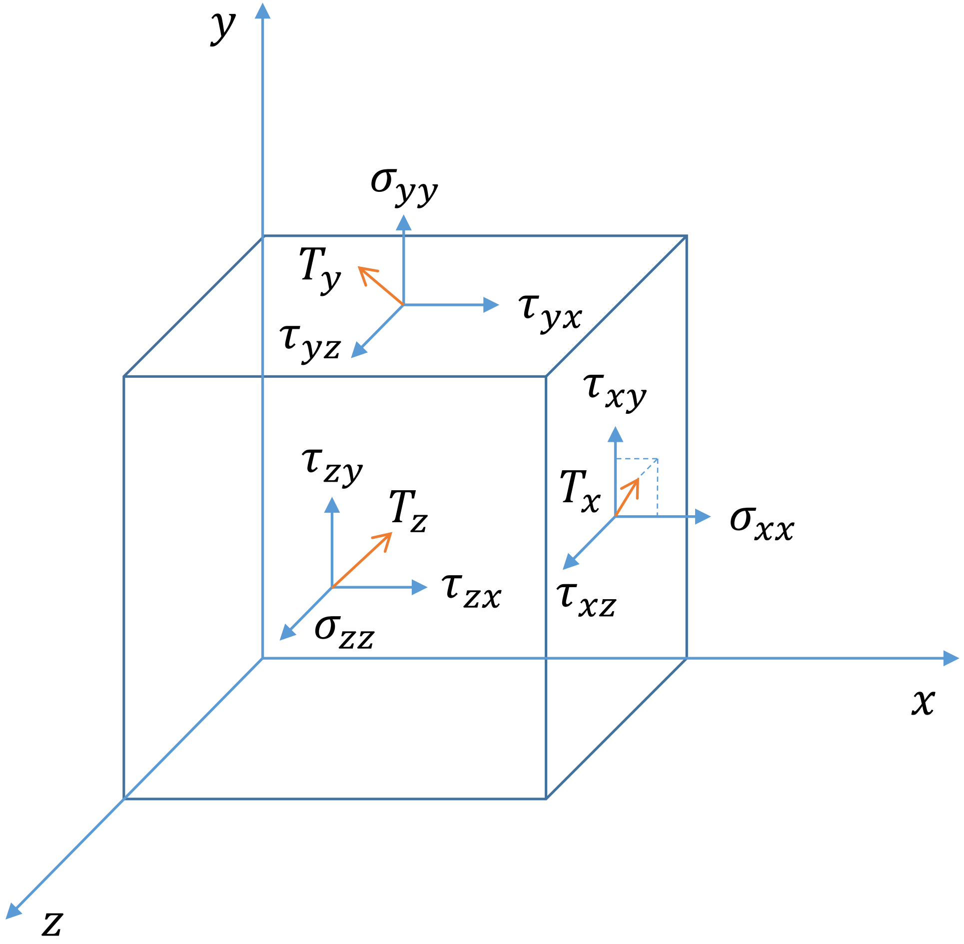
The total number of indices required to identify each component uniquely is equal to the dimension of the array, and is called the order, degree or *rank* of the tensor

- 0th order tensors – scalars (magnitude, 1 value)
- 1st order tensors – vectors (magnitude + direction, e.g. 3 values in 3D space)
- 2nd order tensors – dyads (quantities that have magnitude and two directions, represents variation of magnitude, e.g. 3×3 values)
- 3rd order tensors – triads (e.g. 3×3×3 values)
- 4rd order tensors – Einstein's general relativity required a tensor of rank 4 (x, y, z, t), i.e. 4×4×4×4 = 256 components
- and so on…
- Tensors (together with scalars and vectors) are important quantities in many fields: mechanics, electrodynamics, fluid mechanic, crystal structure, general relativity and others
- But the visualization of tensors is quite challenging

Graphical Representation of Dyad

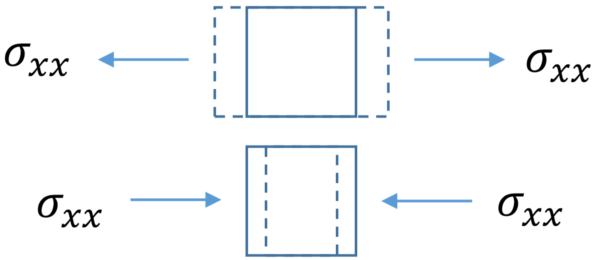
$$\begin{matrix} T_x \\ T_y \\ T_z \end{matrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

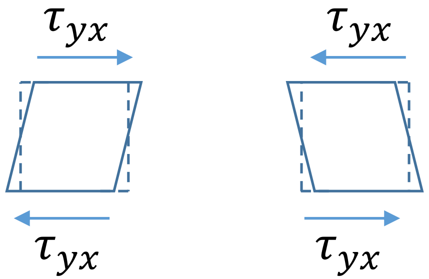


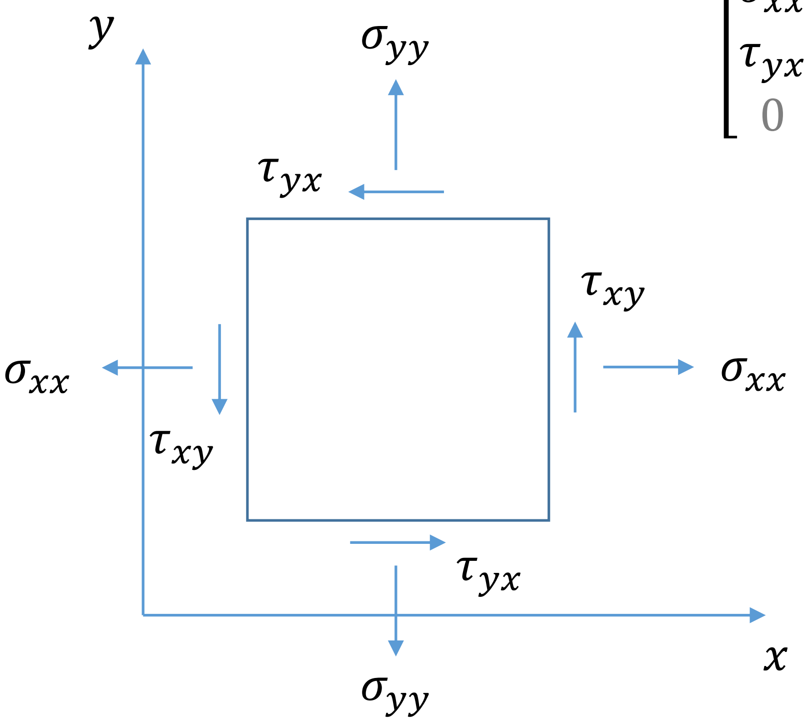
Stress Tensor in 2D

Force acting along the axis perpendiculary to the sides ($\sigma = \frac{F_{\text{normal}}}{A}$) causes stretching ($\sigma_{xx} > 0$ or $\sigma_{yy} > 0$) or compression ($\sigma_{xx} < 0$ or $\sigma_{yy} < 0$)



Force acting along the sides ($\tau = \frac{F_{\text{parallel}}}{A}$) causes shear stress ($\tau_{xy} \neq 0$ or $\tau_{yx} \neq 0$)





$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_{xx} = \frac{F_x}{A_x}$$

$$\sigma_{yy} = \frac{F_y}{A_y}$$

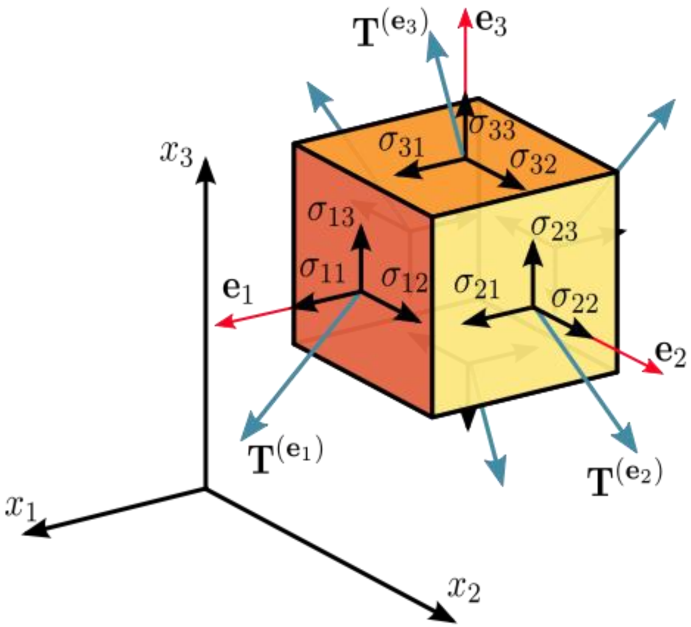
$$\tau_{xy} = \frac{F_y}{A_x}$$

$$\tau_{yx} = \frac{F_x}{A_y}$$

Stress Tensor in 3D

- Cauchy stress tensor – 2nd order tensor
- Completely define the state of stress at a point inside a material

$$\begin{bmatrix} T_1^{(\mathbf{n})} & T_2^{(\mathbf{n})} & T_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}.$$



Second Order Tensors

- May be defined as an operator that acts on a vector $\boldsymbol{u}$ generating another vector $\boldsymbol{v}$, so that $\mathbf{T}(\boldsymbol{u}) = \boldsymbol{v}$ or $\mathbf{T} \cdot \boldsymbol{u} = \boldsymbol{v}$
- $\mathbf{T}$ is a mapping that takes one vector as input, and gives one vector as output

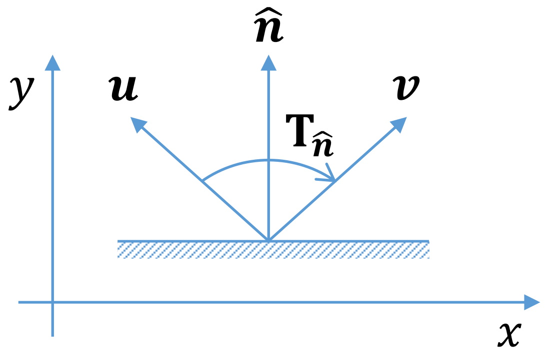
  $\mathbf{T}: V \rightarrow V$, where V is a vector space
- It is a linear operator (linear transformation) so it holds that $\mathbf{T}$ is
  - distributive $\mathbf{T}(\boldsymbol{a} + \boldsymbol{b}) = \mathbf{T}\boldsymbol{a} + \mathbf{T}\boldsymbol{b}$
  - associative $\mathbf{T}(k\boldsymbol{a}) = k\mathbf{T}(\boldsymbol{a})$

  for all $\boldsymbol{a}, \boldsymbol{b} \in V$ and $k \in \mathbb{R}$
- Tensors $\mathbf{S}$ and $\mathbf{T}$ are equal iff $\mathbf{S} \cdot \boldsymbol{v} = \mathbf{T} \cdot \boldsymbol{u}$ for any $\boldsymbol{u}, \boldsymbol{v} \in V$

Second Order Tensor Example

- Operator which transforms every vector into its mirror-image with respect to a given plane

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \mathbf{T}_{\hat{\boldsymbol{n}}} \cdot \boldsymbol{u} = \mathbf{T}_{\hat{\boldsymbol{n}}}(\boldsymbol{u}) = \boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

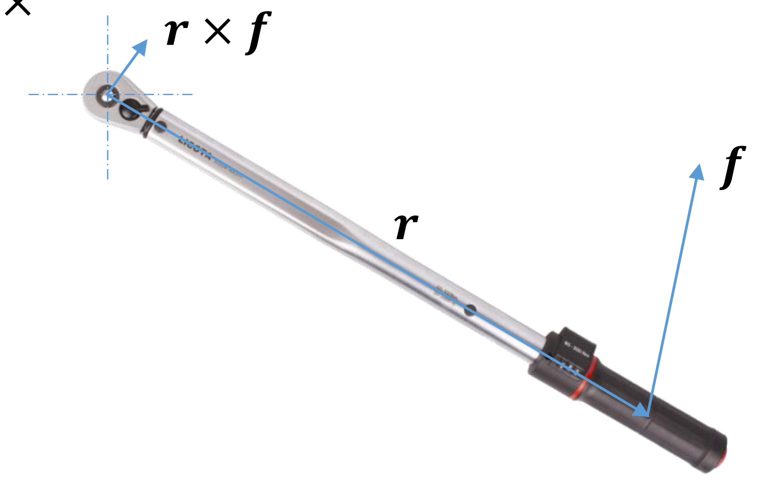


- Note that the situation will be the same in 3D

Second Order Tensor Example

- Operator which transforms force vector $\boldsymbol{f}$ into the moment/torque vector $\boldsymbol{r} \times \boldsymbol{f}$

$$\underbrace{\begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}}_{\boldsymbol{r}\times} \cdot \boldsymbol{f} = \mathbf{T}_r \cdot \boldsymbol{f} = \mathbf{T}_r(\boldsymbol{f}) = \boldsymbol{r} \times \boldsymbol{f}$$



Tensor Product (Dyad)

- The tensor product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is written as $\boldsymbol{u} \otimes \boldsymbol{v}$
  - Dot product $\boldsymbol{u} \cdot \boldsymbol{v}$
  - Cross product $\boldsymbol{u} \times \boldsymbol{v}$
  - Direct sum $\boldsymbol{u} \oplus \boldsymbol{v}$
  - Tensor (outer) product $\boldsymbol{u} \otimes \boldsymbol{v}$

„making new vectors from old“

$$\boldsymbol{u} \oplus \boldsymbol{v} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}$$

$$\boldsymbol{u} \otimes \boldsymbol{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 v_1 \\ u_1 v_2 \\ u_2 v_1 \\ u_2 v_2 \\ u_3 v_1 \\ u_3 v_2 \end{bmatrix} \overset{\text{reshape}}{\Leftrightarrow} \begin{bmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \\ u_3 v_1 & u_3 v_2 \end{bmatrix} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$$

1st order tensor with dimension 3

1st order tensor with dimension 2

2nd order tensor with dimensions (3, 2)

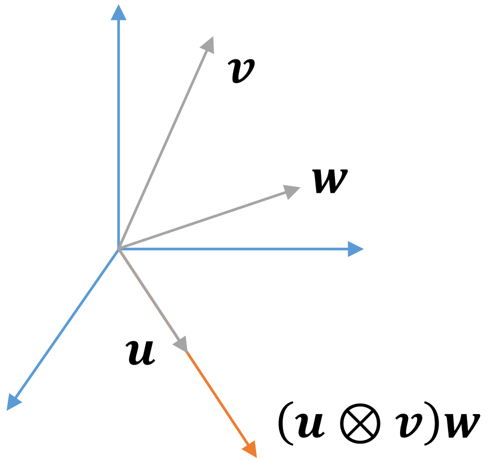
$\mathbb{R}^m \otimes \mathbb{R}^n$ is isomorphism to $\mathbb{R}^{mn}$

Tensor Product (Dyad)

- Tensor product of two vectors (dyad transformation) can be defined as follows

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} = \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w})$$

- It transforms a vector $\boldsymbol{w}$ into a new vector with the direction of $\boldsymbol{u}$ and length of $\|\boldsymbol{u}\|(\boldsymbol{v} \cdot \boldsymbol{w})$



Let us have two vectors $u := \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $v := \begin{bmatrix} 7 \\ 6 \\ -5 \end{bmatrix}$

and if their tensor (outer) product is

$$u \cdot v^{\mathrm{T}} = \begin{bmatrix} 7 & 6 & -5 \\ -14 & -12 & 10 \\ 21 & 18 & -15 \end{bmatrix}$$

then for an arbitrary vector w, e.g. $w := \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$

the following equality must hold

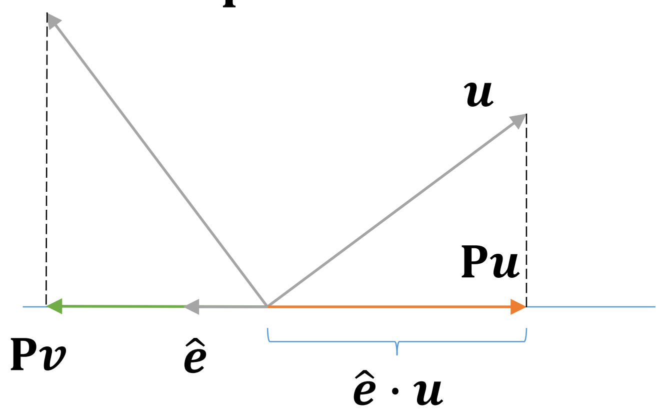
$$\left(u \cdot v^{\mathrm{T}}\right) \cdot w = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} \quad = \quad u \cdot (v \cdot w) = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

- Note that the tensor product is not commutative

Projection Tensor

- Consider the dyad $\hat{\boldsymbol{e}} \otimes \hat{\boldsymbol{e}}$ then from the definition we get

$$\underbrace{(\hat{\boldsymbol{e}} \otimes \hat{\boldsymbol{e}})}_{\mathbf{P}}\boldsymbol{u} = \hat{\boldsymbol{e}}(\hat{\boldsymbol{e}} \cdot \boldsymbol{u})$$



- $\hat{\boldsymbol{e}} \otimes \hat{\boldsymbol{e}}$ is called projection tensor

Tensor Data Visualization

- Scalar field, e.g. $s\colon \mathbb{R}^3 \to \mathbb{R}$
- Vector field, e.g. $\boldsymbol{u}\colon \mathbb{R}^3 \to \mathbb{R}^3$
- Tensor field, e.g. $\mathbf{T}\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

- With tensor fields, we measure some magnitude at some point and in some direction
- Tensors are independent of the coordinate systems

Tensor Data Visualization

- Tensor attributes are high-dimensional generalization of vectors and matrices
- In other words, tensor data encode some spatial property that varies as a function of position and direction
- A tensor with rank (order) r in a n-dimensional space has r indices and n^r components (r is a number representing simultaneous directions)
  - Scalars ($r = 0$) … n^0 component with no index (value)
  - Vectors ($r = 1$) … n^1 components with only one index (vector[i])
  - Matrices ($r = 2$) … n^2 components with two indices (matrix[i][j])
  - Tensors ($r \geq 3$)

Only where $\frac{\partial f}{\partial x}(x) = 0$, otherwise $k = \frac{y''}{(1+y'^2)^{3/2}}$

$x_0: f(x) = 0, k(x) = \frac{\partial^2 f}{\partial x^2}(x) = 0$

$x_1: f(x) = -x^2, k(0) = \frac{\partial^2 f}{\partial x^2}(0) = -2$

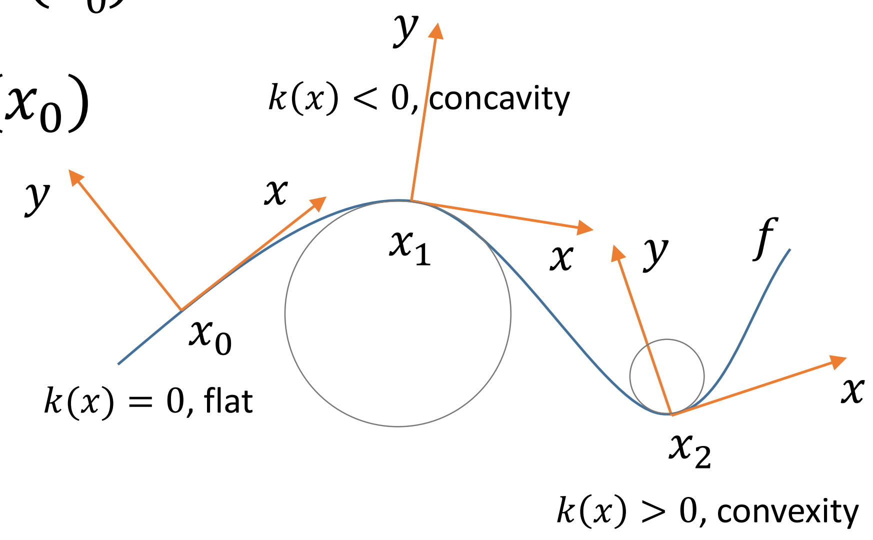
$x_2: f(x) = 2x^2, k(0) = \frac{\partial^2 f}{\partial x^2}(0) = 4$

Tensor Data Visualization

- Example: curvature of a planar curve
  - In local xy coordinate system, a curve can be described as $y = f(x)$ (explicit curve) in the neighborhood of a point x_0 where $f(x_0) = 0$
  - Signed curvature is then defined as $k(x) = \frac{\partial^2 f}{\partial x^2}(x_0)$

    a single number (pointing to $k(x)$)

  - Alternative definitions of curvature
    - $k(x) = 1/\text{radius of circle tangent to } f$ at x_0
    - How quickly normal $\boldsymbol{n}$ changes around x_0



Curvature of Plane Curves

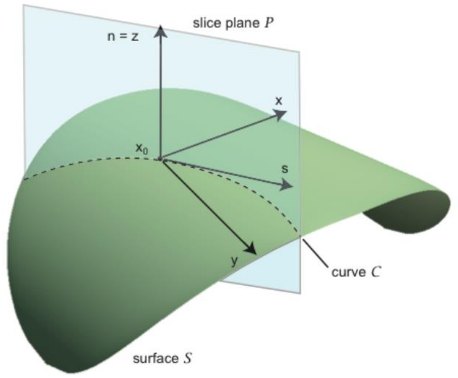
- Let $c(t) = (x(t), y(t))$ be a proper ($\mathrm{d}c/\mathrm{d}t$ is defined, differentiable and nowhere equal to the zero vector) parametric representation of a plane curve then the signed curvature is

$$k = \frac{x'y'' - y'x''}{\left(x'^2 + y'^2\right)^{3/2}}$$

- Example: $x(t) = r\cos(t) + \ x_0; \mathrm{y}(t) = r\sin(t)\ +\ x_0$

$$k = \frac{r\ \sin(t)\ r\ \sin(t) + r\ \cos(t)\ r\ \cos(t)}{(r^2\ \sin^2(t) + r^2\ \cos^2(t) + \ )^{3/2}} = \cdots = \frac{r^2}{r^3} = \frac{1}{r}$$

Tensor Data Visualization



- Example: curvature of a surface
  - In local xyz coordinate system, a surface can be described as $\mathrm{z} = f(x, y)$ (explicit surface) in the neighborhood of a point x_0 where $f(x_0) = 0$
  - Definition of a curvature is analogous with planar case but in which direction to look for changes? This implies that the curvature of a surface at some point cannot be descibed with a single number
  - We have to compute $C(x_0) = \frac{\partial^2 f}{\partial \boldsymbol{s}^2}(x_0) = \boldsymbol{s}^T H \boldsymbol{s}$ where H is called the Hessian of f

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

The problem is that we need to construct local coordinate systems at every point on surface and it is not obvious how to do that

and $\boldsymbol{s}$ is the direction in which we look for the surface (normal) curvature

Tensor Data Visualization

- Example: curvature of a surface
  - Solution is to let the surface be describe by an implicit function $f(x, y, z) = 0$
  - Then we can express the curvature of the surface f as

$$C(x_0, \boldsymbol{s}) = \frac{\partial^2 f}{\partial \boldsymbol{s}^2}(x_0) = \frac{\boldsymbol{s}^T H \boldsymbol{s}}{\|\nabla f(x_0)\|},$$

  where H is the 3×3 Hessian matrix in global coordinate system

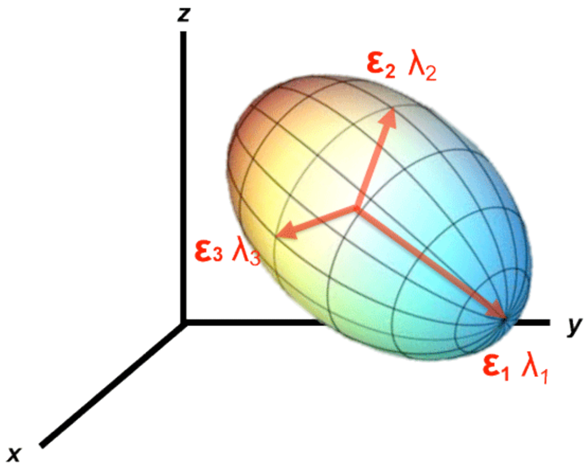
$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

A curvature tensor of the given surface f is fully described by 3×3 matrix of 2nd order derivatives

Recall that $\mathbf{T}$: $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

Tensor Data Visualization

- Tensor fields are common quantity in engineering and physical sciences:
  - Stress, strain, diffusion, velocity gradients, etc.
- Mostly second-order tensors – interpreted as a linear transformation between vectors (represented in 3D by 3x3 matrices):
  - Stress to strain, force to deformation
- Special case – symmetric second-order tensors:
  - Can be viewed as anisotropic ellipsoids (eigenvectors and eigenvalues are principal axes of the diffusion ellipsoids)



Tensor Data Visualization

- Example: diffusion tensor
  - Consider an anisotropic material
  - We have to compute diffusivity at a point **x** in a direction **s**

$$D(\mathbf{x},\mathbf{s}) = \frac{\partial^2 f}{\partial \mathbf{s}^2}(\mathbf{x})$$

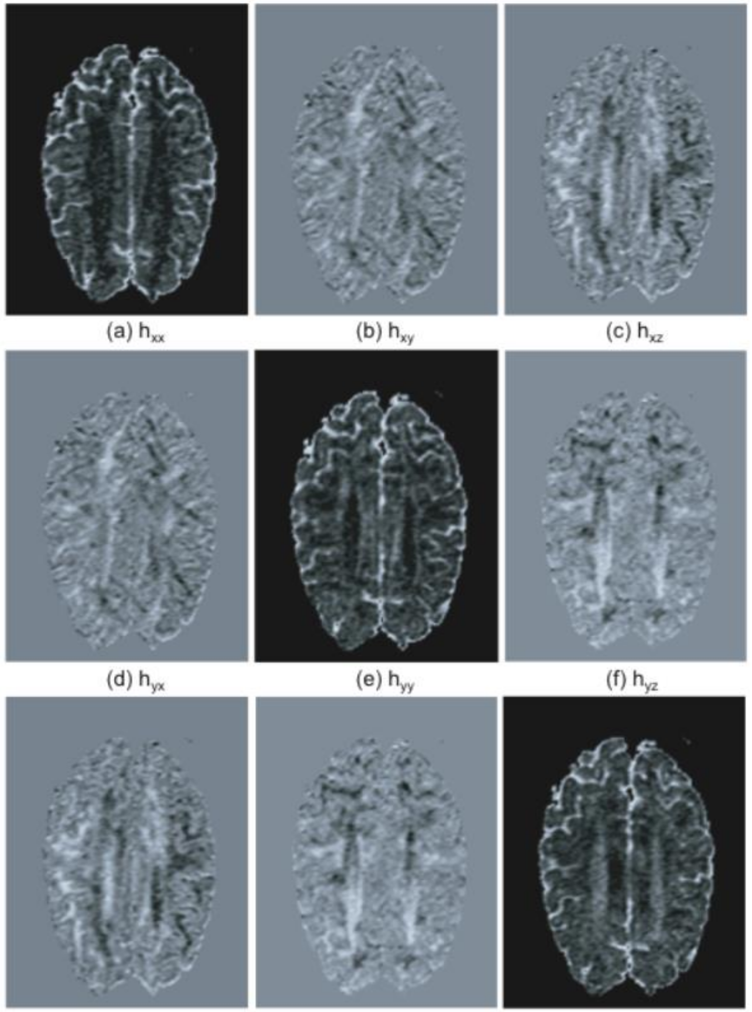
f … speed of water motion in tissue

  - Application: Diffusion of water in the human brain tissue
    - Strong along neural fibers
    - Weak across fibers

Tensor Data Visualization

- Example: diffusion tensor
  - Compute Hessian in R^3
  - Select some slice of interest
  - Visualize all components of H using color mapping

  - We get 9 images, some are same due to the symmetry
  - But we do not really care about diffusion along x, y, z axes



Tensor Data Visualization

- Principal component analysis

α … angle of ***s*** with local coordinate axis $\mathbf{x}_0$

$$\boldsymbol{s}^{\mathrm{T}} = (\cos(\alpha), \sin(\alpha))$$

$$C(\mathbf{x},\mathbf{s}) = \frac{\partial^2 f}{\partial \mathbf{s}^2}(\mathbf{x}) = \mathbf{s}^T \mathbf{H}\mathbf{s} = h_{11}\cos^2\alpha + (h_{12}+h_{21})\sin\alpha\cos\alpha + h_{22}\cos^2\alpha$$

We are looking for extremal curvature

$$\frac{\partial C}{\partial \alpha}(\mathbf{x},\mathbf{s}) = 0 \rightarrow \quad -h_{11}\cos\alpha\sin\alpha - \frac{h_{12}+h_{21}}{2}(\sin^2\alpha - \cos^2\alpha) + h_{22}\sin\alpha\cos\alpha = 0$$

$$\begin{aligned} h_{11}\cos\alpha + h_{12}\sin\alpha &= \lambda\cos\alpha \\ h_{21}\cos\alpha + h_{22}\sin\alpha &= \lambda\sin\alpha. \end{aligned}$$

- This is equivalent to a system of equations in matrix form

$$H\mathbf{s} = \lambda\mathbf{s}$$

Note that the matrix H only stretches (eigen) vector ***s***

$$(H - \lambda I)\mathbf{s} = 0$$

From linear algebra, this is equivalent to $\det(H - \lambda I) = 0$

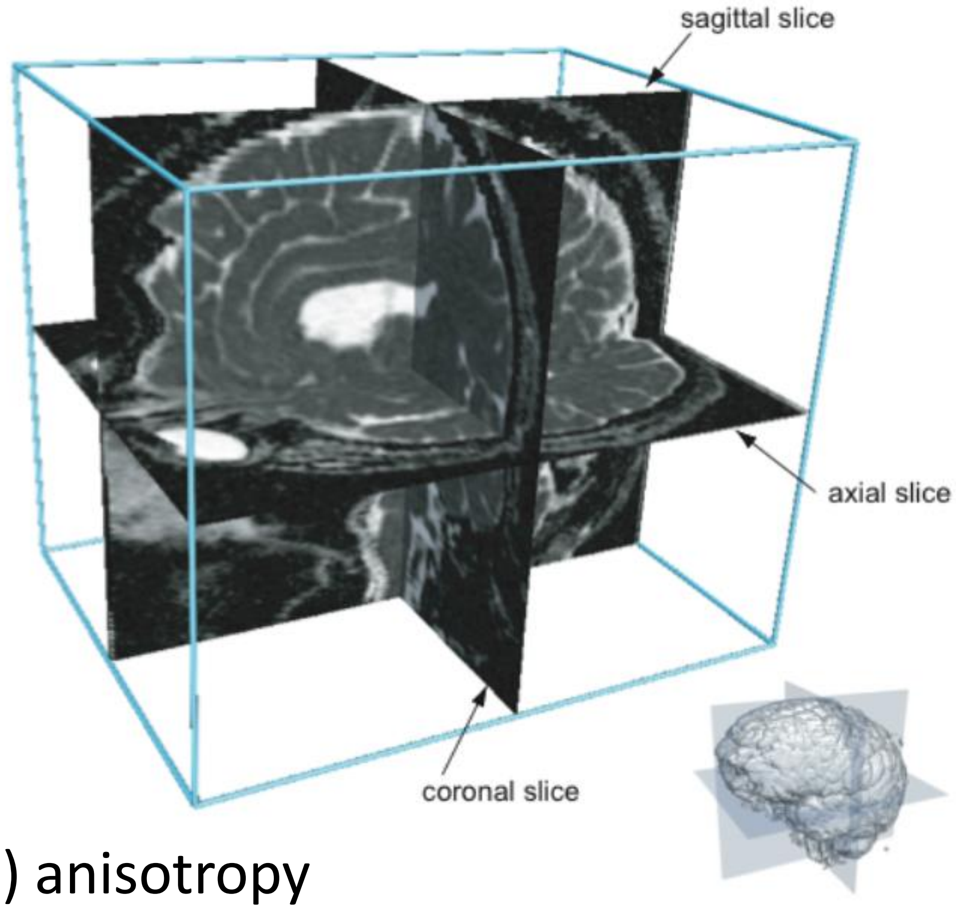
Tensor Data Visualization

- Principal component analysis
- Mean diffusivity

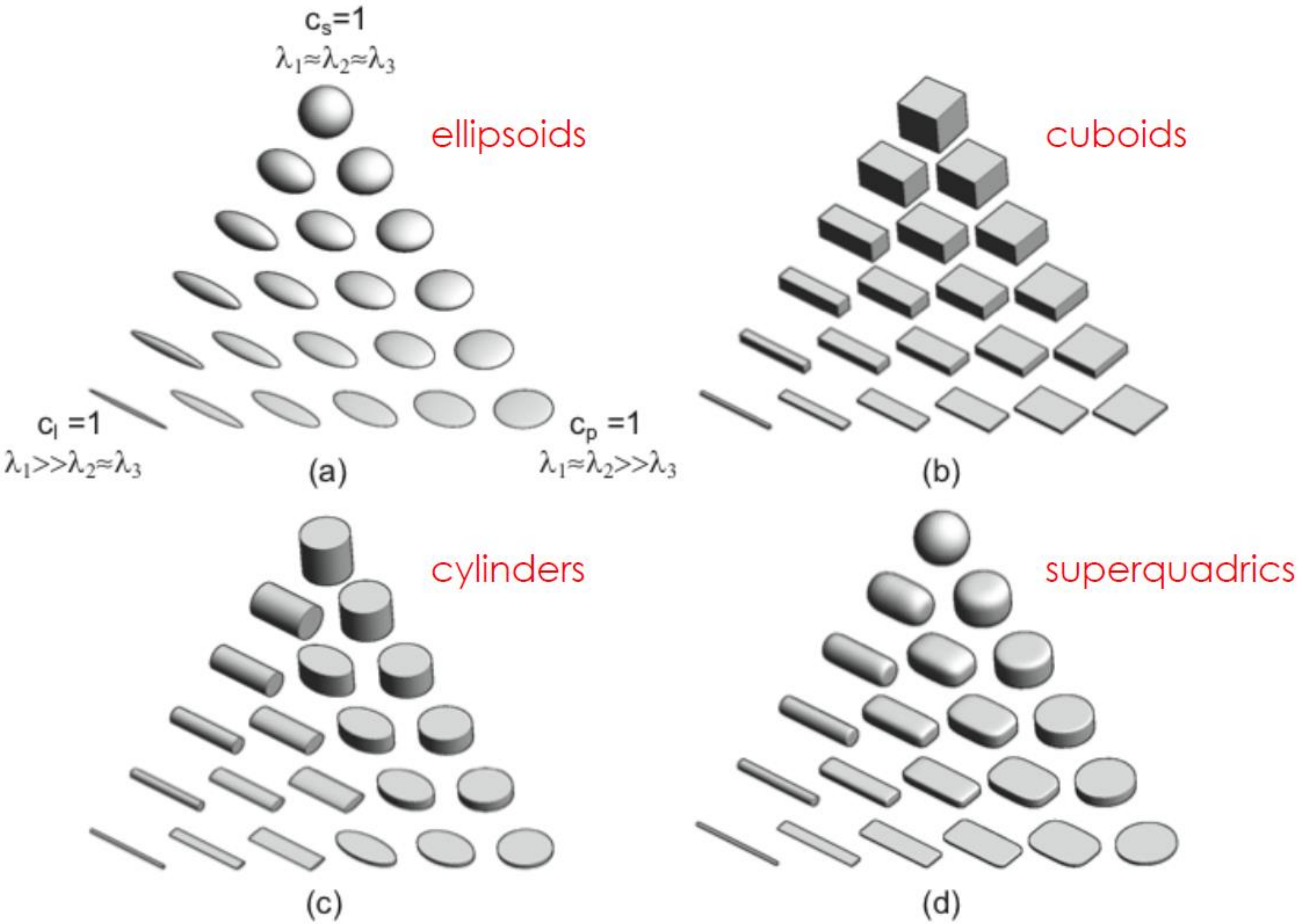
$$\mu = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3)$$

- White = strong mean diff.
- Black = weak mean diff.

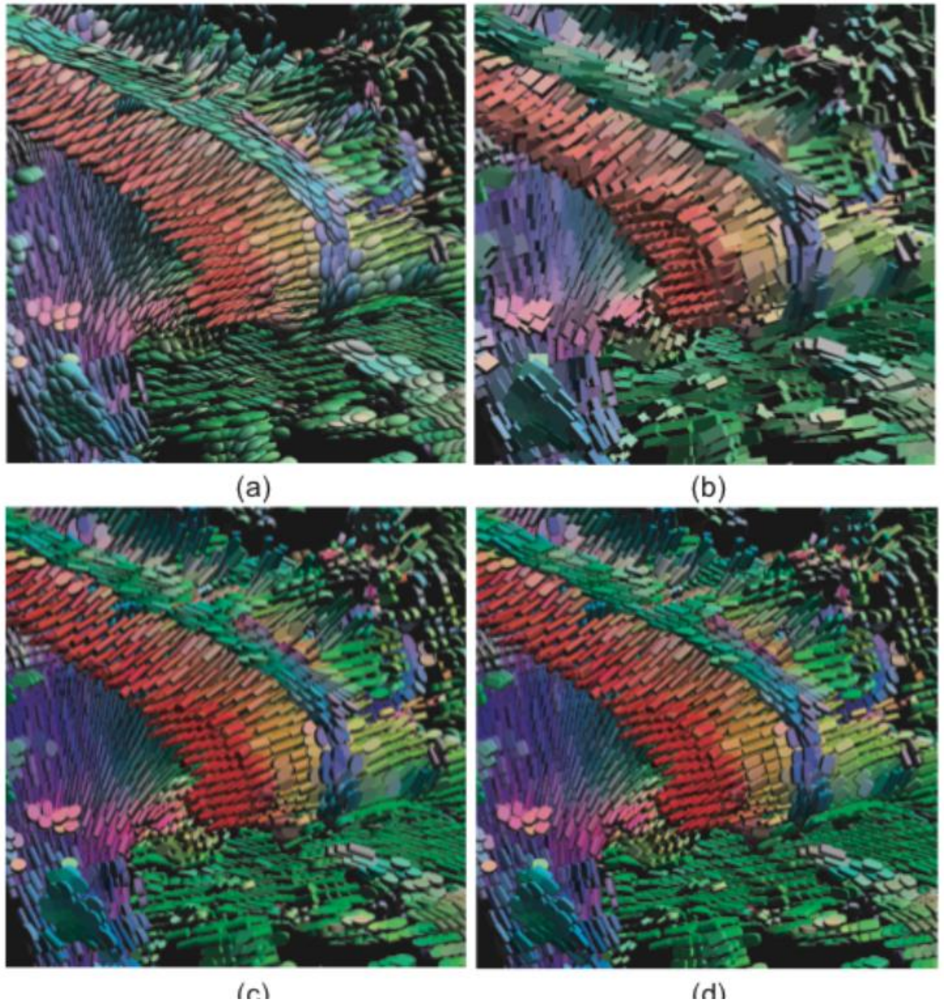
- Other measures: (fractional/relative) anisotropy



Tensor Data Visualization



Tensor Data Visualization



a) ellipsoids
b) cuboids
c) cylinders
d) superquadrics

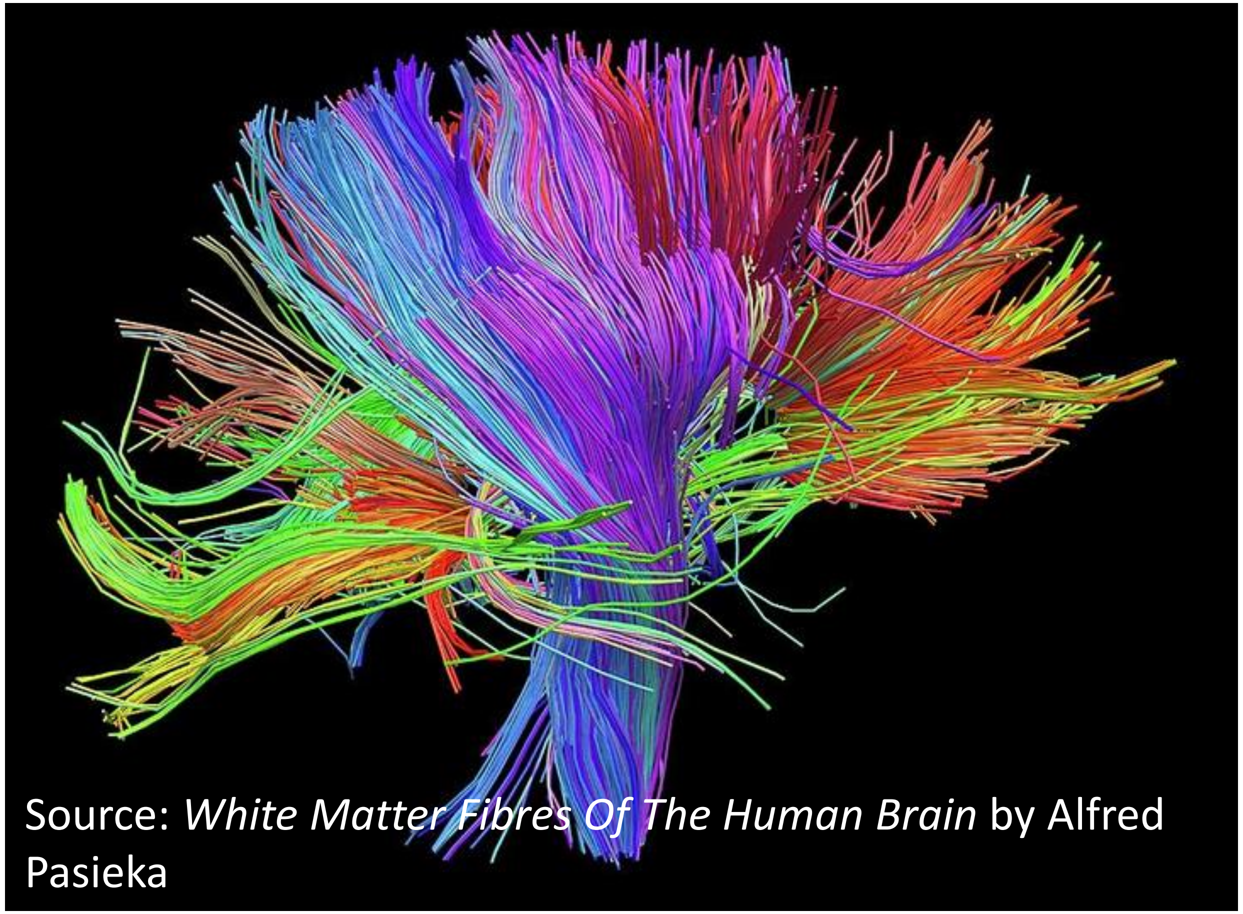
Superquadrics look arguably most ‘natural’

Tensor Data Visualization

- Tensor visualization:
  - Component visualization
  - Anisotropy visualization
  - Major eigenvector visualization
- Fiber tracking
  - Basic fiber tracking
  - Stream tubes
  - Hyperstreamlines

Fiber Tracking

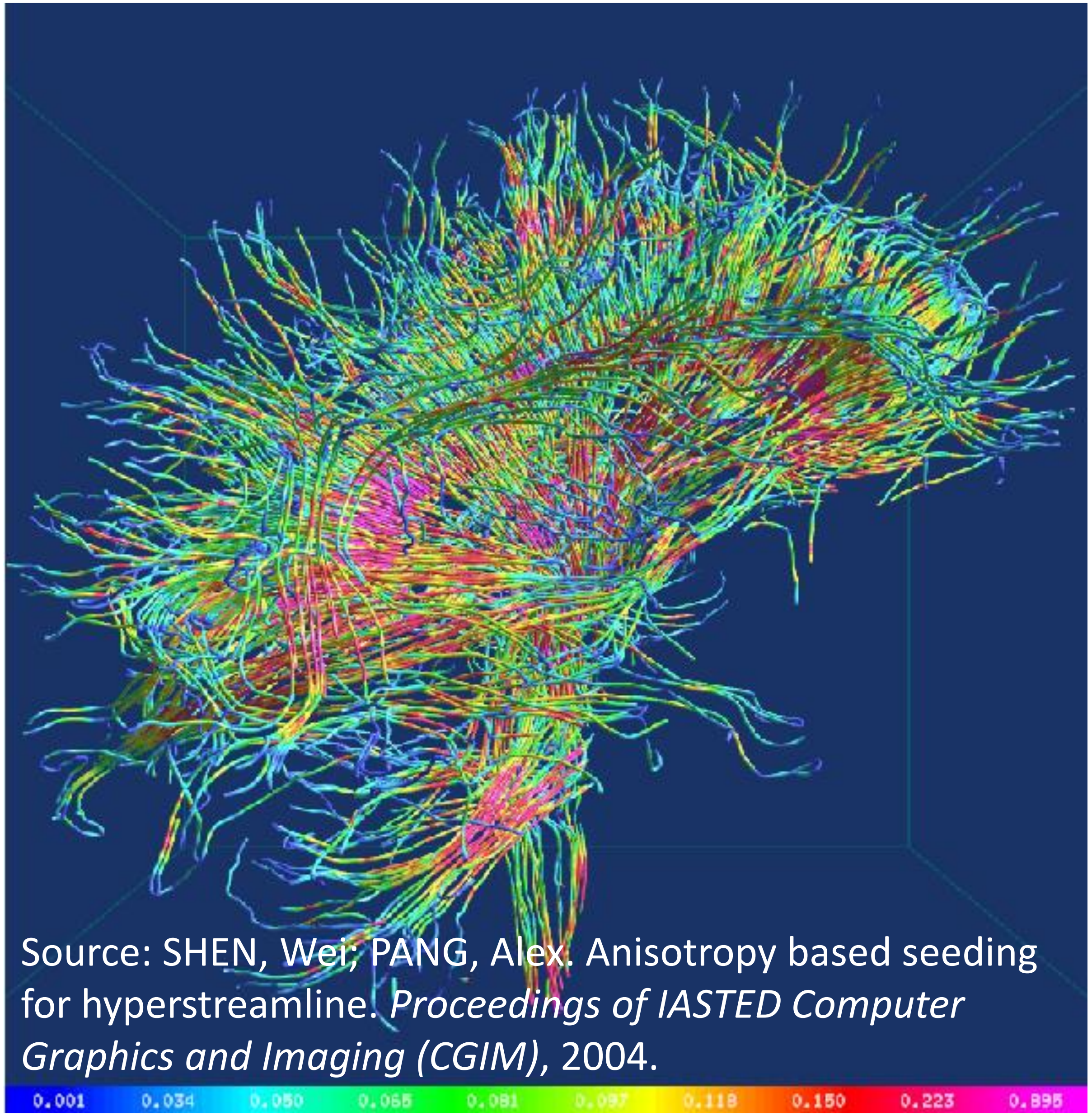
- Similar to streamlines in case of vector fields visualization
- Tracks the direction of the major eigenvectors
- Tubes have circular crossection
- Color indicates the local direction of the hyperstreamlines
- Mostly used for DT-MRI tensors



Source: *White Matter Fibres Of The Human Brain* by Alfred Pasieka

Hyperstreamlines

- Extension of fiber tracking that enables us to visualize direction information from the tensor field beyond the major eigenvector
- Construct stream tubes in the direction of major eigenvector
- Circular cross section is replaced by an elliptical cross section controlled by medium and minor eigenvector



Source: SHEN, Wei; PANG, Alex. Anisotropy based seeding for hyperstreamline. *Proceedings of IASTED Computer Graphics and Imaging (CGIM)*, 2004.